UNIT - 2

SPECIAL FUNCTIONS

0) Exponential
1) Gamma
2) Beta
3) Bessel's

Improper Integral

- integral with one limit as $\pm\infty$ or containing a position where function is discontinuous
- eg: $\int_0^{\infty}$, $\int_{-\infty}^{a}$, $\int_0^1 \frac{1}{x}\,dx$ or $\int_{-1}^{1} \frac{1}{x}\,dx$

Gamma Function

For any $n > 0$, gamma function is denoted by $\Gamma(n)$ and is defined as

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\,dx$$

DIFFERENT FORMS OF GAMMA FUNCTION

1) $$\Gamma(n) = \int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^{n-1} dx$$

$t = \ln\frac{1}{x} \Rightarrow x = e^{-t}$

$dx = -e^{-t}\,dt$

$x = 0,\ t \to \infty$

$x = 1,\ t \to 0$

$$\Gamma(n) = \int_{\infty}^{0} t^{n-1} - e^{-t} dt$$

$$\Gamma(n) = \int_{0}^{\infty} e^{-t} t^{n-1} dt \longrightarrow \text{Gamma function.}$$

2) $$\Gamma(n) = 2\int_{0}^{\infty} e^{-x^2} x^{2n-1} dx.$$

$t=0, x=0 \quad t=x^2 \quad dt = 2x\,dx.$
$t \to \infty, x \to \infty \quad x^{2n-1} = x^{2n-2} \cdot x = t^{n-1} \cdot x$

$$= \int_{0}^{\infty} e^{-t} t^{n-1} dt.$$

$$\Gamma(n) = \int_{0}^{\infty} e^{-t} t^{n-1} dt$$

Properties of Gamma Function.

1. $\Gamma(n+1) = n\Gamma(n).$

$$\Gamma(n) = \int_{0}^{\infty} e^{-x} x^{n-1} dx$$

$$\Gamma(n+1) = \int_{0}^{\infty} e^{-x} x^{n} dx = \int_{0}^{\infty} \cancel{x e^{-x} x^{n-1}} dx$$

$\cancel{u = x}$ $\quad$ $V = \cancel{\int e^{-x} x^{n-1}} dx$

$\cancel{du = dx}$ $\quad$ $dv = e^{-x} \cancel{x^{n-1}} dx$

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n dx$$

$u = x^n$ $\quad$ $V = -e^{-x}$

$du = nx^{n-1} dx$ $\quad$ $dv = e^{-x} dx$

$$= \left[-x^n e^{-x}\right]_0^\infty + \int_0^\infty e^{-x} n x^{n-1} dx \longrightarrow (1)$$

$$\left(-x^n e^{-x}\right)_0^\infty = \lim_{x \to \infty} -x^n e^{-x}$$

$$= \lim_{x \to \infty} \frac{-x^n}{e^x} \quad \text{form } \frac{\infty}{\infty}$$

Applying L'Hôpital's Rules n times

$$= \lim_{x \to \infty} \frac{n!}{e^x} = 0.$$

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$\therefore$ (1) reduces to

$$\Gamma(n+1) = n \int_0^\infty e^{-x} x^{n-1} \, dx$$

$$\Gamma(n+1) = n\,\Gamma(n)$$ ← similar to factorial where $(x+1)! = (x+1)\,x!$

2. $\Gamma(n+1) = n!$

$$\Gamma(n+1) = n\,\Gamma(n)$$
$$= n(n-1)\,\Gamma(n-2)$$
$$\vdots$$
$$= n(n-1)(n-2)\cdots(1)\,\Gamma(1)$$

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = \left[-e^{-x}\right]_0^\infty$$
$$= \left(\frac{-1}{e^x}\right)_0^\infty = 1$$

$$\therefore \Gamma(n+1) = n(n-1)(n-2)\cdots(1)$$
$$= n!$$ for any +ve integer

$$\Gamma(n+1) = n!$$

Note: Just like how log tables exist, Γ tables exist

1. Find $\left(\frac{1}{2}\right)!$

$$\left(\frac{1}{2}\right)! = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2} x^{2\left(\frac{1}{2}\right)-1}\,dx = 2\int_0^\infty e^{-x^2}\,dx$$

$$\text{let } I = \int_0^\infty e^{-x^2}\,dx = \int_0^\infty e^{-y^2}\,dy$$

$$I^2 = \int_0^\infty e^{-x^2}\,dx \int_0^\infty e^{-y^2}\,dy$$

$$= \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

converting to polar

$$I^2 = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\,dr\,d\theta$$

$t = r^2, \quad dt = 2r\,dr$

$$= \frac{1}{2}\int_0^{\pi/2}\int_0^{\infty} e^{-t}\,dt\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left[-e^{-t}\right]_0^{\infty} d\theta$$

$$I^2 = \frac{1}{2}\int_0^{\pi/2} 1\,d\theta = \frac{\pi}{4}.$$

$$\therefore I = \frac{\sqrt{\pi}}{2}$$

$$\therefore \Gamma\left(\frac{1}{2}\right) = 2\int_0^{\infty} e^{-x^2}\,dx = 2\frac{\sqrt{\pi}}{2} \Rightarrow \boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

$$\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\boxed{\therefore \left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}}$$

2. What is $\left(-\frac{1}{2}\right)!$?

$$\left(-\frac{1}{2}\right)! = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

3. $\Gamma(-1/2) = ?$

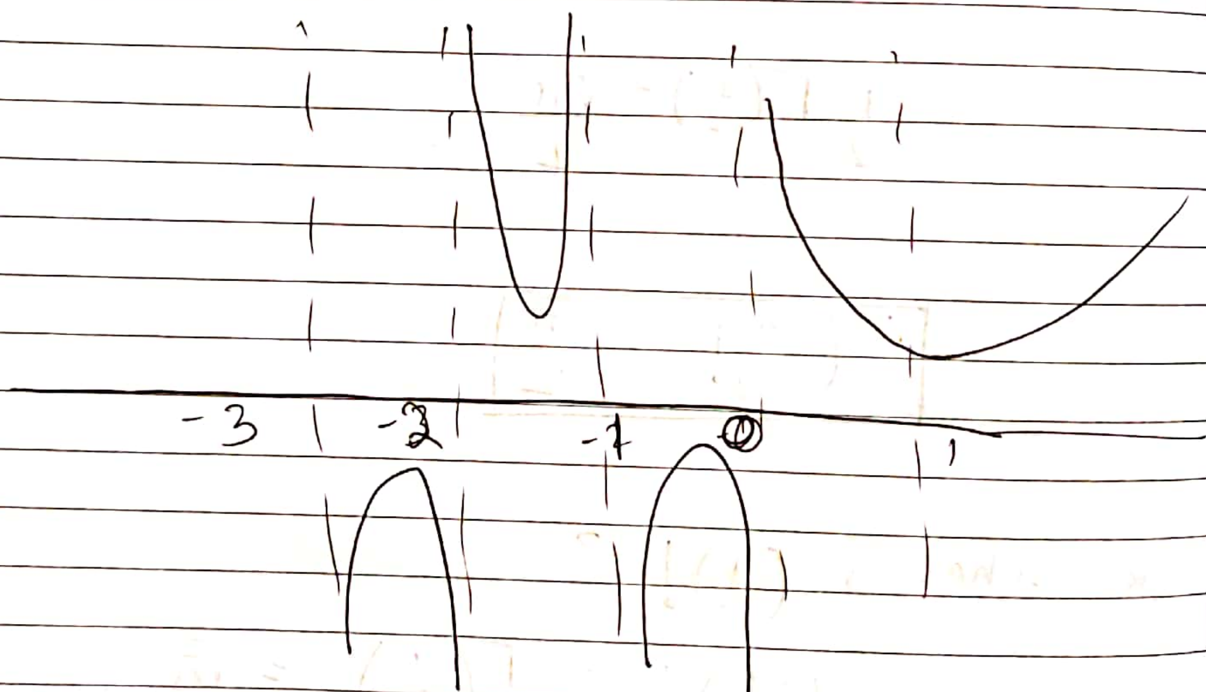
$\cancel{\Gamma(-1/2)}$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$\Gamma(-1/2) = \frac{\Gamma(1/2)}{-1/2} = -2\sqrt{\pi}$$

$$\Gamma(-1/2) = -2\sqrt{\pi}$$

Graph of Gamma Function



Beta Functions

If m and n are positive, then

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

DIFFERENT FORMS OF $\beta(m,n)$

1). Trigonometric form

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta.$$

Proof

LHS: $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

let $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\,d\theta.$

$x = 0,\ \theta = 0$

$x = 1,\ \theta = \pi/2$

$$\int_0^{\pi/2} \sin^{2m-2}\theta \, \cancel{(1-\sin^2\theta)} \cos^{2n-2}\theta \; 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta = \text{RHS}$$

Note: $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

2) Improper integral form

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

Proof

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

① let $x = \frac{1}{1+y}$ $\quad dx = \frac{-1}{(1+y)^2} dy$

$x = 0, \; y \to \infty$
$x = 1, \; y = 0$

①

$$= \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1} \left(\frac{y}{y+1}\right)^{n-1} \left(\frac{-1}{(1+y)^2}\right) dy$$

$$= \int_0^{\infty} \left(\frac{1}{1+y}\right)^{m-1} \left(\frac{y}{1+y}\right)^{n-1} \left(\frac{1}{(1+y)^2}\right) dy$$

$$= \int_0^{\infty} \frac{(y^{n-1})}{(1+y)^{m-1}(1+y)^{n-1}(1+y)^2} dy$$

$$= \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

(2) $x = \frac{y}{1+y}$ $\quad dx = \frac{(1+y) - y}{(1+y)^2} dy = \frac{1}{(1+y)^2} dy$

$x = 0, \quad y = 0$

$x = 1, \quad y \to \infty$

$$= \int_0^{\infty} \left(\frac{y}{1+y}\right)^{m-1} \left(\frac{1}{1+y}\right)^{n-1} \left(\frac{1}{1+y}\right)^2 dy$$

$$= \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}} dy$$

Properties of β

1. $\beta(m,n) = \beta(n,m)$

Proof:

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$$

Let $x = 1-y$ $\quad$ $x=0,\ y=1$

$dx = -dy$ $\quad$ $x=1,\ y=0$

$$= \int_1^0 (1-y)^{m-1} y^{n-1} (-dy)$$

$$= \int_0^1 y^{n-1} (1-y)^{m-1}\, dy = \beta(n,m)$$

2. Relation between β and Γ

$$\beta(m,n) = \frac{\Gamma m\, \Gamma(n}{\Gamma(m+n)}$$

Q Proof:

take $\beta(m,n)\ \Gamma(m+n) = \Gamma(m)\ \Gamma(n)$

RHS: $\Gamma(m)\Gamma(n)$

$$= 2\int_0^\infty e^{-x^2} x^{2m-1}\, dx \cdot 2\int_0^\infty e^{-y^2} y^{2n-1}\, dx$$

$$= 4\int_0^\infty \int_0^\infty e^{-x^2-y^2} x^{2m-1} y^{2n-1}\, dx\, dy$$

$x = r\cos\theta$ $\quad dx\,dy = r\,dr\,d\theta$

$y = r\sin\theta$

$$= 4\int_0^\infty \int_0^{\pi/2} e^{-r^2} r^{2m-1} (\cos\theta)^{2m-1} r^{2n-1} (\sin\theta)^{2n-1}\, r\,dr\,d\theta$$

$$= 4\int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} r^{2(m+n)-1} \cos^{2m-1}\theta \sin^{2n-1}\theta\, dr\,d\theta$$

$$= 2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta \cdot 2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\, dr$$

$= \beta(n,m) \cdot \Gamma(m+n)$ Hence proved.

4. Using the result, ~~pt~~

$$\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}, \text{ prove that}$$

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

$$\Gamma(m)\Gamma(n) = \Gamma(m+n)\,\beta(n,m)$$

$$\Gamma(n+1-n) = \Gamma(1) = 1.$$

$$\therefore \Gamma(n)\Gamma(1-n) = \beta(n,1-n)$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)}\,dx = \int_0^\infty \frac{x^{n-1}}{1+x}\,dx$$

$$= \frac{\pi}{\sin n\pi} \quad \text{(given)}$$

Hence proved.

Formulas

1. $$\beta(m,n) = \beta(n,m) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = 2\int_0^{\pi/2} \sin^{2m-1}x \cos^{2n-1}x\,dx.$$

2. $$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

3. $$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi} = \beta(n, 1-n)$$

4. $$\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

(taking $m=0$)

5. $$\int_0^{\pi/2} \sin^n x = \int_0^{\pi/2} \cos^n x = \frac{1}{2}\beta\left(\frac{1}{2}, \frac{n+1}{2}\right)$$

6. $$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx = \int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^{n-1}dx$$

$$= 2\int_0^\infty e^{-x^2}x^{2n-1}\,dx$$

7. $\Gamma(1/2) = \sqrt{\pi}$ $\quad$ $\Gamma(-1/2) = -2\sqrt{\pi}$

Legendre's Duplication Formula

$$\Gamma(n)\,\Gamma(n+1/2) = 2^{1-2n}\,\Gamma(2n)\sqrt{\pi} \longrightarrow (1)$$

or

$$\beta(n,n) = 2^{1-2n}\,\beta(n, 1/2)$$

(1) $\Gamma(n)\,\Gamma(n+1/2)\,2^{2n-1} = \Gamma(2n)\,\Gamma(1/2)$

~~Proof~~ $\cancel{\Gamma(n)\,\Gamma(n+1/2)}$

$$\beta(n,n) = \frac{\Gamma(n)\,\Gamma(n)}{\Gamma(2n)}$$

$$\beta(n, 1/2) = \frac{\Gamma(n)\,\Gamma(1/2)}{\Gamma(n+1/2)}$$

Proof: $\beta(n,n) = 2\int_0^{\pi/2} \sin^{2n-1}x \cos^{2n-1}x \, dx$

$$= 2\int_0^{\pi/2} \left(\frac{\sin 2x}{2}\right)^{2n-1} dx$$

$$= 2 \cdot 2^{1-2n} \int_0^{\pi/2} (\sin 2x)^{2n-1} \, dx$$

$2x = t$ $\quad x : 0 \to \pi/2$

$2dx = dt$ $\quad t : 0 \to \pi$

$$= 2^{1-2n} \int_0^{\pi} \sin^{2n-1} t \, dt$$

$\sin(\pi - \theta) = \sin\theta$

$$= 2 \cdot 2^{1-2n} \int_0^{\pi/2} \sin^{2n-1} t \, dt$$

$$= 2^{1-2n} \cdot \left\{ 2 \int_0^{\pi/2} \sin^{2n-1} t \cos^0 t \, dt \right\}$$

$$\beta(n,n) = 2^{1-2n} \beta(1/2, n)$$

$$\frac{\Gamma(n)\,\Gamma(n)}{\Gamma(2n)} = \beta(n,n) = 2^{1-2n} \beta(1/2, n)$$

$$\frac{\Gamma(n)\cancel{\Gamma(n)}}{\Gamma(2n)} = 2^{1-2n} \frac{\Gamma(1/2)\cancel{\Gamma(n)}}{\Gamma(1/2 + n)}$$

$$\Gamma(n)\Gamma(1/2 + n) = 2^{1-2n} \Gamma(2n)\Gamma(1/2)$$

$$\Gamma(n)\Gamma(1/2 + n)\, 2^{2n-1} = \Gamma(2n)\sqrt{\pi}$$

Hence proved

$$\int_0^\infty x^p e^{-ax^q}\,dx = \frac{\Gamma\left(\frac{p+1}{q}\right)}{\left(a^{\frac{p+1}{q}}\right) q}$$

5. Prove that $\Gamma\left(\frac{p+1}{q}\right) = q\, a^{\frac{p+1}{q}} \int_0^\infty x^p e^{-ax^q}\,dx$

~~LHS:~~

$$\cancel{\Gamma\left(\frac{p+1}{q}\right) = \int_0^\infty e^{-x} x^{\left(\frac{p+1}{q}-1\right)}\,dx}$$

$$\cancel{= \int_0^\infty e^{-x} x^{\left(\frac{p+1-q}{q}\right)}\,dx}$$

RHS:

$$q\, a^{\frac{p+1}{q}} \int_0^\infty x^p e^{-ax^q}\,dx$$

$x = \left(\frac{y}{a}\right)^{1/q}$ $\qquad y = +ax^q$

$\qquad dy = +aq\,x^{q-1}\,dx$

$dx = \frac{dy}{aq}$

$\therefore$ $x = 0,\; y = 0$

$x \to \infty,\; y \to \infty$

$$= q\, a^{\frac{p+1}{q}} \int_0^\infty$$

$$= q\, a^{\frac{p+1}{q}} \int_0^\infty \left(\frac{+y}{a}\right)^{p/q} \frac{e^{-y}\,dy}{aq\,x^{q-1}}$$

$$= \int_0^\infty \frac{a^{\frac{p+1}{q}} a^{1/q} (+y)^{p/q}}{a^{(p/q)}} \frac{e^{-y}\,dy}{a\cdot\left(\frac{y}{a}\right)^{\frac{(q-1)}{q}}}$$

$$= a^{1/q-1} \int_0^\infty \frac{y^{p/q} e^{-y} a^{(p-\frac{p}{q})}}{(y)^{(p-\frac{p}{q})}} dy$$

6. Prove that $\Gamma(n+1) = (m+1)^{n+1} (-1)^n \int_0^1 x^m (\ln x)^n dx$.

RHS:

$$(m+1)^{n+1} (-1)^n \int_0^1 x^m (\ln x)^n dx$$

$x=0, t \to \infty \quad -t = \ln x \quad \Rightarrow x = e^{-t}$

$x=1, t=0 \quad -dt = \frac{dx}{x} \quad dx = -e^{-t} dt$

$$(m+1)^{n+1} (-1)^n \int_\infty^0 -e^{-mt} (-t)^n e^{-t} dt$$

$$= (m+1)^{n+1} (-1)^n \int_0^\infty e^{-t(m+1)} (-t)^n dt$$

$$= (m+1)^{n+1} \int_0^\infty e^{-(m+1)t} t^n dt$$

$y = (m+1)t$

$dy = (m+1) dt$

$t = \left(\frac{y}{m+1}\right)^n$

$$= (m+1)^n \int_0^\infty e^{-y} \left(\frac{y}{m+1}\right)^n \frac{dy}{}$$

$$= \int_0^\infty e^{-y} y^n \, dy = \int_0^\infty e^{-(y)} y^{(n+1-1)} \, dy$$

$$= \Gamma(n+1)$$

$$\boxed{\int_0^1 x^m (\ln x)^n \, dx = \frac{(-1)^n n!}{(m+1)^{n+1}}}$$

7. Evaluate the following integrals.

$$\int_0^\infty e^{-y^2} \sqrt{y} \, dy = I$$

$$\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} \, dx.$$

$n = 3/4$

$$\Gamma\left(\frac{3}{4}\right) = 2\int_0^\infty e^{-x^2} x^{1/2} \, dx$$

$$I = \frac{\Gamma(3/4)}{2}$$

8 Evaluate $\int_0^\infty 3^{-4x^2}\, dx = I$

$$= \int_0^\infty e^{\ln(3^{-4})x^2}\, dx = \int_0^\infty e^{-4\ln 3 x^2}\, dx$$

$$t = +(4\ln 3)x^2 \quad \Rightarrow \quad x = \frac{t^{1/2}}{(4\ln 3)^{1/2}}$$

$$dt = +8\ln 3\, x\, dx$$

$$= \int_0^\infty e^{-t} \frac{t^{-1/2}\, dt\,(4\ln 3)^{1/2}}{\cancel{(4\ln 3)}(8\ln 3)} \cdot \frac{(4\ln 3)^{1/2}}{\cancel{(4\ln 3)}(8\ln 3)} \int_0^\infty e^{-t}\, t^{1/2-1}\, dt$$

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx \quad (n = 1/2)$$

$$\Gamma(1/2) = \int_0^\infty e^{-t}\, t^{-1/2}\, dt = \sqrt{\pi}$$

$$\boxed{I = \cancel{\frac{\Gamma(3/2)}{16(\ln 3)^{3/2}}}} \qquad \cancel{\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}}$$

$$I = \frac{\sqrt{\pi}}{32(\ln 3)^{3/2}}$$

$$I = \frac{2\sqrt{\ln 3}}{8\ln 3}\sqrt{\pi} = \frac{\sqrt{\pi}}{4\sqrt{\ln 3}}$$

$$\boxed{I = \frac{\sqrt{\pi}}{4\sqrt{\ln 3}}}$$

9. $$\int_0^1 \frac{dx}{\sqrt{-\ln x}} = \int_0^1 \frac{dx}{\sqrt{(\ln 1/x)}}$$

$$= \int_0^1 (\ln 1/x)^{-1/2}\, dx \qquad n-1 = -1/2, \quad n = 1/2$$

$$= \Gamma(1/2) = \boxed{\sqrt{\pi}}$$

10. $$\int_0^1 x^4 (1-x)^3\, dx.$$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

$$m = 5,\ n = 4.$$

$$= \beta(5,4) = \frac{\Gamma(5)\Gamma(4)}{\Gamma(9)} = \frac{4! \times 3!}{8!}$$

$$= \frac{\cancel{3}\times\cancel{2}\times 1}{8\times 7\times\cancel{6}\times 5} = \frac{1}{280}$$

11. $$\int_0^2 x\sqrt[3]{(8-x^3)}\,dx$$

$$\cancel{8-x^3 = y^3}$$

$$= \int_0^2 x\sqrt[3]{8\left(1-\left(\frac{x}{2}\right)^3\right)}\,dx$$

$$= 2\int_0^2 x\left(1-\left(\frac{x}{2}\right)^3\right)^{1/3} dx. \qquad t = x/2, \quad dt = dx/2$$

$$= 4\int_0^1 2t\,(1-t^3)^{1/3}\,dt.$$

$$t^3 = y$$
$$3t^2\,dt = dy$$
$$dt = \frac{dy}{3y^{2/3}}$$

$$= \frac{8}{3}\int_0^1 y^{-1/3}(1-y)^{1/3}\,dy$$

$$= \cancel{\frac{8}{3}\int_0^1} \quad \frac{8}{3}\beta\left(\frac{2}{3},\frac{4}{3}\right) = \frac{8}{3}\,\frac{\Gamma(2/3)\,\Gamma(4/3)}{\Gamma(2)}$$

$$= \frac{8\times 1\,\Gamma(2/3)\,\Gamma(1/3)}{3\cdot 3} = \frac{8}{9}\,\frac{\pi}{\sin\frac{\pi}{3}} = \frac{8\pi\times}{9\sqrt{3}}$$

$$= \boxed{\frac{16\pi}{9\sqrt{3}}}$$

12. $\beta\left(\frac{1}{3}, \frac{2}{3}\right) = \beta(n, 1-n) = \frac{\pi}{\sin n\pi}$

$= \frac{\pi}{\sin \pi/3} = \frac{2\pi}{\sqrt{3}}$

13. $\beta(2.5, 1.5) = \cancel{\frac{\Gamma(2.5)\Gamma(1.5)}{\Gamma(4)}}$

$\cancel{\Gamma(2.5} = \frac{\Gamma(2.5)\Gamma(1.5)}{\Gamma(4)} = \frac{\Gamma(2.5)\Gamma(1.5)}{3!}$

$\Gamma(n+1) = n\Gamma(n)$

$= \frac{(1.5)\Gamma(1.5)\,\Gamma(1.5)}{\Gamma(4)}$

$= \frac{(1.5)(0.5)\Gamma(0.5)\,0.5\,\Gamma(0.5)}{6}$

$= \frac{3}{2} \times \frac{1}{2} \times \frac{1}{2} \times \left(\Gamma(1/2)\right)^2 \times \frac{1}{6}$

$\beta(2.5, 1.5) = \frac{\pi}{16}$

14. $\Gamma(-15/2) = \frac{\Gamma(-13/2)}{-15/2} = \frac{\Gamma(-11/2)}{\frac{-15}{2} \times \frac{-13}{2}}$

$= \frac{\Gamma(+1/2)}{\frac{-15}{2} \times \frac{-13}{2} \times \frac{-11}{2} \times \frac{-9}{2} \times \frac{-7}{2} \times \frac{-5}{2} \times \frac{-3}{2} \times -1/2}$

$$= \frac{256\sqrt{\pi}}{2027025} = 2.24\times10^{-4}$$

15. $\cancel{\int_0^2 x\sqrt[3]{8-x^3}\,dx}$ $\quad \int_0^{2\pi} \sin^8\theta\, d\theta$

$$= 4\int_0^{\pi/2} \sin^8\theta\, d\theta = 4\times\frac{1}{2}\beta\left(\frac{1}{2}, \frac{8+1}{2}\right)$$

$$= 2\beta\left(\frac{1}{2}, \frac{9}{2}\right) = \frac{2\,\Gamma(1/2)\Gamma(9/2)}{\Gamma(5)}$$

$$= \frac{\cancel{2}}{4\times\cancel{3}\times\cancel{2}} \times \Gamma(1/2)\times 7/2\times 5/2\times 3/2\times 1/2\,\Gamma(1/2)$$

$$= \frac{35\pi}{64}$$

16. $\int_0^{\pi/2} \sin^4\theta \cos^5\theta\, d\theta = \int_0^{\pi/2} \sin^{3+1}\theta \cos^{4+1} d\theta$

$$= \frac{1}{2}\beta\left(\frac{5}{2}, \frac{6}{2}\right) = \frac{1}{2}\frac{\Gamma(5/2)\Gamma(3)}{\Gamma(11/2)} = \cancel{\frac{4! \times 5!}{40!}}$$

$$= \cancel{\frac{1}{1260}} \qquad = \frac{1}{2}\frac{\cancel{3}\times 2\times \cancel{\frac{3}{2}}\times \cancel{\frac{1}{2}\times\sqrt{\pi}}}{9/2\times 7/2\times 5/2\times \cancel{3/2}\times\cancel{1/2\sqrt{\pi}}}$$

$$= \frac{\cancel{3}\times\cancel{2}\times 8\times\frac{1}{\cancel{2}}}{\cancel{3}9\times 7\times 5} = \frac{8}{315} = \frac{8}{315}$$

17. $$\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \int_0^{\pi/2} \cos^{1/2}\theta \, \sin^{-1/2}\theta \, d\theta$$

$$= \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

$$= \frac{1}{2}\,\frac{\pi}{\sin \pi/4} = \frac{\pi}{\sqrt{2}}$$

18. $$\int_0^{\pi/2} \left(\sqrt{\tan\theta} + \sqrt{\sec\theta}\right) d\theta$$

$$= \int_0^{\pi/2} \sin^{1/2}\theta \, \cos^{-1/2}\theta \, d\theta + \int_0^{\pi/2} \cos^{-1/2}\theta \, d\theta$$

$$= \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right) + \frac{1}{2}\beta\left(\frac{1}{2}, \frac{1}{4}\right)$$

$$= \frac{1}{2}\,\frac{\pi}{\sin \pi/4} + \frac{1}{2}\,\frac{\Gamma(1/2)\,\Gamma(1/4)}{\Gamma(3/4)}$$

$$= \boxed{\frac{\pi}{\sqrt{2}} + \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(1/4)}{\Gamma(3/4)}} = \cancel{\frac{\pi}{\sqrt{2}} + \frac{\sqrt{\pi}}{2}\cdots}$$

19. Prove that $\int_0^\infty e^{-x^4} dx = \frac{1}{4} \Gamma\left(\frac{1}{4}\right)$

$x^2 = t \Rightarrow 2x\,dx = dt$

$dx = \frac{dt}{2\sqrt{t}}$.

$$= \int_0^\infty e^{-t^2} \frac{t^{-1/2} dt}{2} = \frac{1}{2}\int_0^\infty t^{-1/2} e^{-t^2} dt.$$

$$\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx \qquad 2n-1 = -\frac{1}{2}$$

$n = 1/4$

$$\Gamma(1/4) = 2\int_0^\infty e^{-t^2} t^{2(1/4)-1} dt$$

$$\boxed{I = \frac{1}{4} \Gamma(1/4)}$$

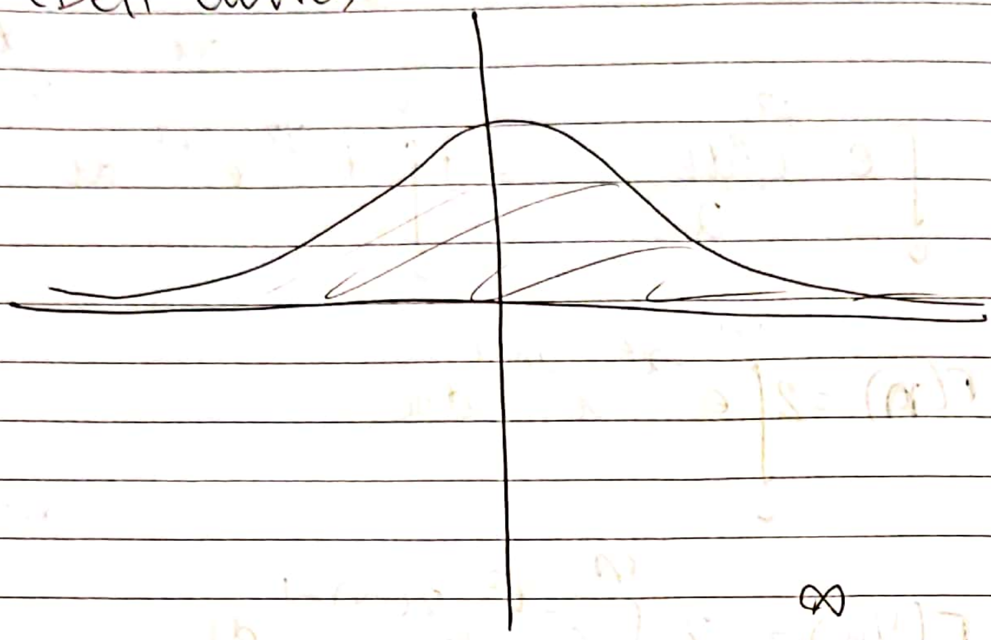
20. Prove that $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \int_0^{\pi/2} (\sin\theta)^{-1/2} d\theta = \pi$

$$= \int_0^{\pi/2} \sin^{1/2}\theta\, d\theta \int_0^{\pi/2} \sin^{-1/2}\theta\, d\theta = \frac{1}{2}\beta\left(\frac{1}{2}, \frac{3}{4}\right) \cdot \times \frac{1}{2}\beta\left(\frac{1}{2}, \frac{1}{4}\right)$$

$$= \frac{1}{\cancel{2}} \frac{\Gamma(1/2)\cancel{\Gamma(3/4)}}{\Gamma(\cancel{5/4})\ 1/4} \times \frac{1}{\cancel{2}} \frac{\Gamma(1/2)\cancel{\Gamma(1/4)}}{\cancel{\Gamma(3/4)}} = \boxed{\pi}$$

21. Show that the area under the normal curve $y = \frac{1}{a\sqrt{2\pi}} e^{-\frac{x^2}{2a^2}}$ and the x-axis is unity.

(Bell curve)



Due to symmetry, area $= 2\int_0^\infty \frac{1}{a\sqrt{2\pi}} e^{-\frac{x^2}{2a^2}}\,dx$

$$= \frac{2}{a\sqrt{2\pi}} \int_0^\infty e^{-\frac{x^2}{2a^2}}\,dx \qquad t = \frac{x^2}{2a^2}$$

$$dt = \frac{\cancel{2}x\,dx}{\cancel{2}a^2}$$

$$= \frac{\cancel{2}}{\cancel{a}\sqrt{\cancel{2}\pi}} \int_0^\infty e^{-t}\, \frac{\cancel{\sqrt{2}}\cancel{a}\,1}{\cancel{x}}\, t^{-1/2}\,dt \qquad x = a\sqrt{2t}$$

$$\cancel{dx = \frac{a\sqrt{2}}{2\sqrt{2t}}}$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t}\, t^{-1/2}\,dt \qquad dx = \sqrt{2}a\,\frac{1}{2}\,t^{-1/2}$$

$$= \frac{1}{\sqrt{\pi}}\,\Gamma(1/2) = 1.$$

22. P.T $\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx = (b-a)^{m+n-1}\beta(m,n)$

let $(x-a) = (b-a)t$ $\quad b-(x+a)-a$

$dx = (b-a)t$

$x = b \rightarrow t = 1$

$x = a \rightarrow t = 0$

$= \int_0^1 (b-a)^{m-1} t^{m-1} \left((b-a) - t(b-a)\right)^{n-1} (b-a)\,dt$

$= (b-a)^{m+n-1} \int_0^1 t^{m-1}(1-t)^{n-1}\,dt$

23. Evaluate $\int_0^{\pi} x \sin^7 x \cos^4 x\,dx = I$

$I = \int_0^{\pi} (\pi - x)\sin^7(\pi-x)\cos^4(\pi-x)\,(dx)$

$2I = \pi \int_0^{\pi} \sin^7 x \cos^4 x\,dx$

$I = \frac{\pi \times 2}{2}\int_0^{\pi/2} \sin^7 x \cos^4 x\,dx$

$= \frac{\pi}{2}\beta\left(4,\frac{5}{2}\right) = \frac{\pi}{2}\frac{\Gamma(4)\Gamma(5/2)}{\Gamma(13/2)}$

$= \frac{\pi}{\cancel{2}} \frac{3\times \cancel{2} \times \cancel{\Gamma(5/2)}}{11/2 \times 9/2 \times 7/2 \times 5/2\, \cancel{\Gamma(5/2)}} = \frac{\cancel{3}\pi \times 16}{11 \times \cancel{9}_3 \times 7 \times 5}$

$$= \frac{16\pi}{1155}$$

24. P.T. $\beta(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \underbrace{\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx}_{I_2}$$

$t = \frac{1}{x}$ $\therefore x = \frac{1}{t}$

$$I_2 = \int_1^0 \frac{t^{1-m}}{\left(1+\frac{1}{t}\right)^{m+n}} \frac{-dt}{t^2}$$

$t: 1$ to 0

$dx = \frac{-dt}{t^2}$

$$= \int_0^1 \frac{t^{1-\cancel{m}+\cancel{m}+n-2}}{(t+1)^{m+n}} dt = \int_0^1 \frac{t^{n-1}}{(t+1)^{m+n}} dt$$

$$\therefore \beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$= \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$$

complete square

25. $\int_a^\infty e^{2ax-x^2}\,dx$

$\cancel{t = 2ax - x^2}$
$\cancel{dt = (2a + 2x)\,dx}$.

$$= \int_a^\infty e^{-(x^2-2ax+a^2)+a^2}\,dx$$

$t = x - a$. $dt = dx$

$$= \int_a^\infty e^{-(x-a)^2} e^{a^2}\,dx = e^{a^2}\int_0^\infty e^{-t^2}\,dt$$

$$= e^{a^2} \times \tfrac{1}{2}\Gamma(1/2)$$

$$= \boxed{e^{a^2} \times \frac{\sqrt{\pi}}{2}}$$

26. $\int_0^\infty x^n e^{-a^2x^2}\,dx$.

$t = a^2x^2 \Rightarrow x = \frac{\sqrt{t}}{a}$
$dt = 2a^2x\,dx$

$dx = \frac{dt}{2\sqrt{t}a}$

$$= \int_0^\infty \frac{t^{n/2-1/2}}{a} e^{-t} \frac{1}{2a}\,dt = \frac{1}{2a^2}\int_0^\infty t^{(n-1)/2} e^{-t}\,dt$$

$$\Gamma(n) = \int_0^\infty e^{-x} x^n\,dx$$

$$= \frac{1}{2a^2}\int_0^\infty e^{-t}\, t^{(n-1)/2}\,dt$$

$$= \frac{1}{2a^2}\Gamma\left(\frac{n-1}{2}\right)$$

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27. $\int_0^a \sqrt{a^n - x^n}\, dx$

28. $\int_0^2 \frac{1}{\sqrt{2x-x^2}}\, dx$

29. $\int_0^\infty \frac{x^a}{a^x}\, dx$

30. P.T. $\int_0^1 \frac{(x^{m-1})(1-x)^{n-1}}{(1+x)^{m+n}}\, dx = \frac{\beta(m,n)}{2^m}$

31. hence evaluate $\int_0^1 \frac{x^3-2x^2+x}{(1+x)^5}\, dx$.

31. P.T $\int_0^1 x^m \left(\ln\left(\frac{1}{x}\right)\right)^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}$

32. P.T $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\, dx \cdot \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}$

33. PT $\int_0^\infty \frac{dx}{(e^{-x}+e^x)^n} = \frac{1}{4}\beta\left(\frac{n}{2},\frac{n}{2}\right)$ and

evaluate $\int_0^\infty \operatorname{sech}^8 x\, dx$

34. P-T. $\int_0^\infty x^{m-1} \cos ax \, dx = \frac{\Gamma(m)}{a^m} \cos\left(\frac{m\pi}{2}\right)$

$\cancel{\text{LHS} = \int_0^\infty x^{m-1} \cos ax \, dx}$

29. $\int_0^\infty \frac{x^a}{a^x} dx = \int_0^\infty x^a e^{-\ln a \, x} dx$ $\quad t = \ln a x$, $dt = \ln a \, dx$

$= \int_0^\infty \frac{t^a}{(\ln a)^{a+1}} e^{-t} dt = \frac{1}{(\ln a)^{a+1}} \Gamma(a+1)$

31. $\int_0^1 x^m \left(\ln\left(\frac{1}{x}\right)\right)^n dx =$ $\quad t = \ln(1/x)$; $dt = -\frac{1}{x} dx$
$x = e^{-t}$; $dx = -e^{-t} dt$
$x = 0, \; t \to \infty$; $x = 1, t = 0$

$= -\int_\infty^0 e^{-mt} t^n e^{-t} dt = \int_0^\infty t^n e^{-(m+1)t} dt$ $\quad y = (m+1)t$, $\frac{dy}{m+1} = dt$

$= \int_0^\infty \frac{y^n}{(m+1)^{n+1}} e^{-y} dy = \frac{1}{(m+1)^{n+1}} \int_0^\infty e^{-y} y^n dy$

$= \frac{\Gamma(n+1)}{(m+1)^{n+1}}$

27. $\int_0^a \sqrt{a^n - x^n} \, dx = a^{n/2} \int_0^a \sqrt{1 - \left(\frac{x}{a}\right)^n} \, dx$ $\quad \left(\frac{x}{a}\right)^n = t$

$x = a t^{1/n}$

$= a^{n/2} \int_0^1 \sqrt{1-t} \, \frac{a}{n} t^{1/n - 1} dt$ $\quad dx = \frac{a}{n} t^{\frac{1}{n}-1} dt$

$= \frac{a^{\left(\frac{n+2}{2}\right)}}{n} \int_0^1 (t^{1/n - 1}) (1-t)^{3/2 - 1} dt = \frac{a^{\frac{n+2}{2}}}{n} \beta\left(\frac{1}{n}, \frac{3}{2}\right)$

$$= \frac{a^{\frac{n+2}{2}}}{n} \beta\left(\frac{1}{n}, \frac{3}{2}\right) = \frac{a^{\frac{n+2}{2}}}{n} \frac{\Gamma(1/n)\left(\Gamma(3/2)\right)}{\Gamma\left(\frac{1}{n}+\frac{3}{2}\right)}$$

$$= \frac{a^{\frac{n+2}{2}}}{n} \frac{\Gamma(1/n)}{\Gamma\left(\frac{1}{n}+\frac{3}{2}\right)} \frac{\sqrt{\pi}}{2}$$

34. $$\int_0^\infty x^{m-1} \cos ax \, dx = \int_0^\infty \cancel{x^{m-1}\left(\frac{e^{iax}+e^{-iax}}{2}\right)} dx$$

$$= \cancel{\frac{1}{2}\left[\int_0^\infty e^{iax} x^{m-1} dx + \int_0^\infty e^{-iax} x^{m-1} dx\right]}$$

$$= \mathrm{Re}\left(\int_0^\infty x^{m-1} e^{-iax} dx\right) \qquad t = iax \Rightarrow x = t/ia, \quad dx = dt/(ai)$$

$$= \mathrm{Re}\left(\int_0^\infty \left(\frac{t}{ai}\right)^{m-1} e^{-t} \frac{dt}{ai}\right) = \mathrm{Re}\left(\frac{1}{(ai)^m} \int_0^\infty t^{m-1} e^{-t} dt\right)$$

$$= \mathrm{Re}\left(\left(\frac{1}{i}\right)^m \frac{1}{a^m} \Gamma(m)\right) = \mathrm{Re}\left(\frac{(-i)^m}{a^m} \Gamma(m)\right)$$

$$(-i)^m = \left(\cos\frac{\pi}{2} \cancel{+} i \sin\frac{\pi}{2}\right)^m = \left(e^{-i\pi/2}\right)^m = e^{-im\pi/2}$$

$$= \mathrm{Re}\left(\frac{e^{-im\pi/2}}{a^m} \Gamma(m)\right) = \boxed{\frac{\Gamma(m)}{a^m} \cos\frac{m\pi}{2}} = \text{RHS}$$

28. $$\int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx = \int_0^2 \frac{dx}{\sqrt{-(x^2-2x+1)+1}} = \int_0^2 \frac{dx}{\sqrt{-(x-1)^2+1}}$$

$t = x-1 \Rightarrow dt = dx$

$$= \int_0^2 \frac{dx}{\sqrt{1-(x-1)^2}} = \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}}$$

$t = \sin\theta$
$dt = \cos\theta\, d\theta$

$$= \int_{-\pi/2}^{\pi/2} \frac{\cos\theta\, d\theta}{\cos\theta} = 2\int_0^{\pi/2} \cos^0\theta\, d\theta = 2 \times \frac{1}{2}\beta\left(\tfrac{1}{2},\tfrac{1}{2}\right)$$

$$= \frac{\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(1)} = \pi$$

33. $$\int_0^\infty \frac{dx}{(e^{-x}+e^{x})^n} = \frac{1}{4}\beta\left(\frac{n}{2},\frac{n}{2}\right)$$

ans

$$\int_0^\infty \frac{e^{-nx}\,dx}{(e^{-2x}+1)^n}$$

$t = e^{-x} \Rightarrow x = -\ln t$

$dx = -\frac{dt}{t}$

$$= -\int_1^0 \frac{t^{n-1}\,dt}{(t^2+1)^n} = \int_0^1 t^{n-1}(1+t^2)^{-n}\,dt$$

$\cancel{y = t^2 \Rightarrow t = \sqrt{y}}$

$\cancel{dt = \frac{dy}{2\sqrt{y}}}$

$$\cancel{\int_0^1 \frac{y^{\frac{n}{2}-\frac{1}{2}-\frac{1}{2}}(1+y)^{-n}}{2}\,dy}$$

~~1+y = s~~

$$= \int_0^1 \frac{y^{n/2-1}(1+y)^{-n}}{2}\,dy$$

$$= \int_0^1 \frac{t^{n-1}}{(1+t^2)^n} dt \qquad t = \tan\theta \quad dt = \sec^2\theta\, d\theta.$$

$$= \int_0^{\pi/4} \frac{(\tan\theta)^{n-1} \sec^2\theta\, d\theta}{(\sec^2\theta)^n} = \int_0^{\pi/4} \frac{(\sin\theta)^{n-1}}{(\cos\theta)^{n-1}} (\cos^2\theta)^{n-1} d\theta$$

$$= \int_0^{\pi/4} \sin^{n-1}\theta \cos^{n-1}\theta\, d\theta \qquad x = 2\theta \quad dx = 2d\theta.$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{n-1}\frac{x}{2} \cos^{n-1}\frac{x}{2} dx = \frac{1}{2^n}\int_0^{\pi/2} \sin^{n-1} x\, dx$$

$$= \frac{1}{2^n} \times \frac{1}{2} \beta\left(\frac{1}{2}, \frac{n}{2}\right)$$

$$= \frac{1}{2^{n+1}} \times \cancel{\frac{2}{2}}\, \beta\left(\frac{n}{2}, \frac{1}{2}\right)$$

we know $\beta(n,n) = 2^{1-2n} \beta(n, 1/2)$ → Legendre's

or $\beta\left(\frac{n}{2} \times \frac{1}{2}\right) = \beta\left(\frac{n}{2}, \frac{n}{2}\right) 2^{n-1}$

LHS

$$\frac{1}{2^{n+1}} \times 2^{n-1} \beta\left(\frac{n}{2}, \frac{n}{2}\right) = \frac{1}{4} \beta\left(\frac{n}{2}, \frac{n}{2}\right)$$

$$\therefore \int_0^{\infty} \frac{dx}{(e^x + e^{-x})^n} = \frac{1}{4} \beta\left(\frac{n}{2}, \frac{n}{2}\right)$$

$$\int_0^\infty \operatorname{sech}^8 x\,dx = \int_0^\infty \left(\frac{2}{e^{-x}+e^{x}}\right)^8 dx$$

$$= 2^8 \int_0^\infty \frac{dx}{(e^{-x}+e^{x})^8} = 2^8 \times \frac{1}{4}\beta(4,4)$$

$$= 2^6 \frac{\Gamma(4)\Gamma(4)}{\Gamma(8)} = \frac{2^6 \times 3!}{7\times\cancel{6}\times5\times4} = \frac{2^4}{35}$$

$$= \frac{16}{35}$$

32. $$\underbrace{\int_0^1 \frac{x^2}{\sqrt{1-x^4}}dx}_{I_1} \underbrace{\int_0^1 \frac{dx}{\sqrt{1+x^4}}}_{I_2} = \frac{\pi}{4\sqrt{2}}$$

$$I_1: \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}}$$

$$x^2 = \sin\theta$$
$$x = (\sin\theta)^{1/2}$$
$$dx = \frac{1}{2}(\sin\theta)^{-1/2}\cos\theta\,d\theta$$

$$I_1 = \frac{1}{2}\int_0^{\pi/2} \frac{(\sin\theta)^{1/2}\cos\theta\,d\theta}{\sqrt{1-\sin^2\theta}} = \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta\,d\theta.$$

$$= \frac{1}{2}\times\frac{1}{2}\,\beta\left(\frac{1}{2},\frac{3}{4}\right)$$

$$= \frac{1}{4}\,\frac{\Gamma(1/2)\Gamma(3/4)}{\Gamma(5/4)}$$

$$I_2 = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$$

$x^2 = \tan\theta$

$x = (\tan\theta)^{1/2}$

$dx = \frac{1}{2}(\tan\theta)^{-1/2}\sec^2\theta\, d\theta$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{(\tan\theta)^{1/2}\sqrt{1+\tan^2\theta}} = \frac{1}{2}\int_0^{\pi/4} \frac{(\sec\theta)}{(\tan\theta)^{1/2}}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{(\cos\theta^{1/2})}{(\sin\theta)^{1/2}(\cos\theta)}\, d\theta = \frac{\sqrt{2}}{2}\int_0^{\pi/4} 2^{-1/2}\sin^{-1/2}\theta\cos^{-1/2}\theta\, d\theta$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/4} \sin^{-1/2}2\theta\, d\theta$$

$t = 2\theta$

$d\theta = \frac{dt}{2}$

$$= \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \sin^{-1/2}t\, dt = \frac{1}{2\sqrt{2}}\times\frac{1}{2}\beta\left(\frac{1}{4},\frac{1}{2}\right)$$

$$= \frac{1}{2\sqrt{2}}\times\frac{1}{2}\,\frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(3/4)}$$

$$\therefore I = \frac{1}{4}\,\frac{\Gamma(1/2)\,\Gamma(3/4)}{1/4\;\Gamma(1/4)}\cdot\frac{1}{2\sqrt{2}}\,\frac{1}{2}\,\frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(3/4)}$$

$$= \frac{\pi}{4\sqrt{2}}$$

LHS.

30. $$\int_0^1 \frac{(x^{m-1})(1-x)^{n-1}}{(1+x)^{m+n}}\,dx = \int_0^1 \frac{1}{1+x}\left(\frac{x}{1+x}\right)^{m-1}\frac{1}{1+x}\left(\frac{1-x}{1+x}\right)^{n-1}dx$$

$$u = \frac{x}{1+x} \quad \text{due} \Rightarrow x = \frac{u}{1-u},\ dx = \frac{du}{(1-u)^2}$$

$$\int_0^{1/2} \cancel{(1-u)}\,(u)^{m-1}\,\cancel{(1-u)}\,(1-2u)^{n-1}\frac{du}{\cancel{(1-u)^2}}$$

$1+x$
$= \frac{1-u+u}{1-u}$
$= \frac{1}{1-u}$

$\frac{1}{1+x} = 1-u$

$\frac{1-x}{1+x}$
$= 1 - \frac{2x}{1+x}$
$= 1-2u$

$$= \int_0^{1/2} u^{m-1}(1-2u)^{n-1}\,du$$

$$t = 2u \qquad u = t/2$$
$$du = dt/2$$

$$= \int_0^1 \frac{t^{m-1}}{2^{m-1}}(1-t)^{n-1}\frac{dt}{2} = \frac{1}{2^m}\beta(m,n) = RHS$$

$$\int_0^1 \frac{x^3-2x^2+x}{(1+x)^5}\,dx = \int_0^1 \frac{x(x^2-2x+1)}{(1+x)^5}\,dx$$

$$= \int_0^1 \frac{x^{2-1}(1-x)^{3-1}}{(1+x)^{2+3}}\,dx = \frac{1}{2^2}\beta(2,3) = \frac{1}{4}\frac{\Gamma(2)\Gamma(3)}{\Gamma(5)}$$

$$= \frac{1}{4}\times\frac{1\times 2!}{4!} = \frac{1}{2}\times\frac{1}{4\times3\times2} = \frac{1}{48}$$

Bessel Functions

- He was studying Keplar's laws.
- Obtained a Laplace equation. $\nabla^2 u = 0$.
- Reduced to ODE

problematic

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0. \longrightarrow (1)$$

Frobenious Method.

Assume $y = C_1 y_1(x) + C_2 y_2(x)$

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} \quad \hookrightarrow \text{assume as solution.}$$

$$x\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (k+r) x^{k+r \cancel{-1+1}}$$

$$x^2 \frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (k+r)(k+r-1)\, x^{k+r \cancel{-2+2}}$$

Substituting into equation (1)

$$\sum_{r=0}^{\infty} a_r (k+r)(k+r-1)\, x^{k+r} + \sum_{r=0}^{\infty} a_r (k+r) x^{k+r}$$

$$+ (x^2 - n^2) \sum_{r=0}^{\infty} a_r x^{k+r} = 0.$$

$\left(\sum_{r=0}^{\infty} a_r x^{k+r}\right)$ is common inside the sum

$$= \sum_{r=0}^{\infty} \left(a_r x^{k+r}\right) \left[\cancel{(k+r)(k+r-1)} (k+r)^2 - (k+r) + (k+r) - n^2 \right]$$

$$+ \sum_{r=0}^{\infty} a_r x^{k+r+2} = 0$$

$$= \sum_{r=0}^{\infty} a_r x^{k+r} \left((k+r)^2 - n^2 \right) + \sum_{r=0}^{\infty} a_r x^{k+r+2} = 0.$$

comparing coefficients of least term (x^k)

$$a_0 (k^2 - n^2) = 0, \quad a_0 \neq 0.$$

$$\boxed{k = \pm n.}$$

Comparing coefficients of x^{k+1} $(r=1)$

$$a_1 \left((k+1)^2 - n^2 \right) = 0$$

$\boxed{a_1 = 0}$ or $2k+1 = 0$, $k = -1/2$ not a polynomial solution.

coeff of x^{k+2}

$$a_2 \left((k+2)^2 - n^2 \right) + a_0 = 0$$

$$a_2 (4k+4) + a_0 = 0.$$

$$\boxed{a_2 = \frac{-a_0}{4(k+1)}}$$

Coeff of x^{k+3}

$$a_3\left((k+3)^2-n^2\right)+a_1=0$$

$$a_3=\frac{-a_1}{(k+3)^2-n^2} \Rightarrow \boxed{a_3=0}$$

(all odd coeff $=0$ or $a_{2r+1}=0$)

Coeff of x^{k+4}

$$a_4\left((k+4)^2-n^2\right)+a_2=0$$

$$a_4=\frac{-a_2}{(8k+16)}=\frac{a_0}{4\cdot 8\cdot(k+1)(k+2)}$$

$$a_4=\frac{a_0}{2^4(k+1)(k+2)\cdot 2!}$$

Coeff of x^{k+6}

$$a_6\left((k+6)^2-n^2\right)+a_4=0$$

$$a_6=\frac{-a_4}{(12k+36)}=\frac{-a_0}{2^6(k+1)(k+2)(k+3)\cdot 3!}$$

In general:

$$a_{2r}=\frac{(-1)^r a_0}{2^{2r}(k+1)(k+2)\cdots(k+r)\,r!}$$

$$a_{2r+1}=0$$

$$y = \sum_{r=0}^{\infty} \frac{(-1)^r a_0 \, x^{k+2r}}{2^{2r} (k+1)(k+2) \cdots (k+r) \, r!}$$

* choose suitable a_0 to simplify $(k+1) \cdots (k+r)$

- if $a_0 = \dfrac{1}{2^k \Gamma(k+1)}$

$$y = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{k+2r}}{\Gamma(k+1)(k+1)(k+2) \cdots (k+r) \, r!}$$

$$y = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{k+2r}}{\Gamma(k+r+1) \, r!}$$

- if $k = n$, we obtain a Bessel function of first kind

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{\Gamma(n+r+1) \, r!}$$

- if $k = -n$, we obtain a Bessel function of first kind

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{-n+2r}}{\Gamma(-n+r+1) \, r!}$$

- $J_n(x)$ and $J_{-n}(x)$ are Bessel functions.

$$G.S \; y = c_1 J_n(x) + c_2 J_{-n}(x)$$

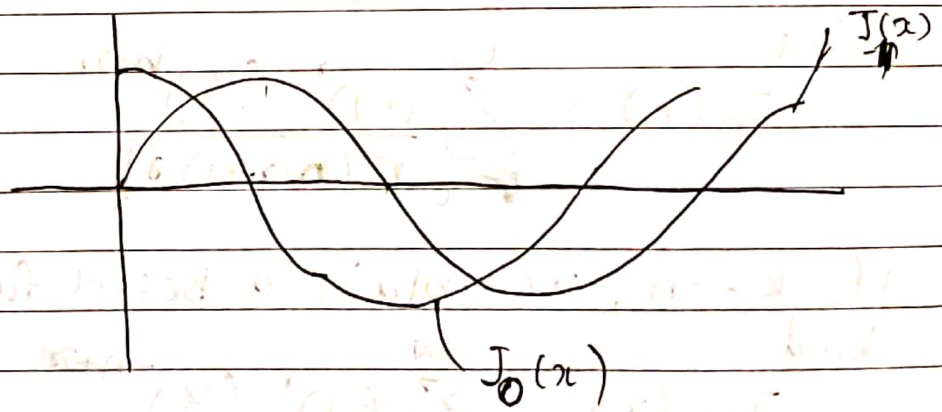
$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{0+2r}}{\Gamma(0+r+1)\, r!}$$

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{2r}}{(r!)^2}$$

$$= 1 + \frac{x^2}{(2^2)(1!)^2} - \frac{x^4}{(2^4)(2!)^2} - \cdots \approx \cos x$$

$$J_1(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{1+2r}}{\Gamma(r+2)\, r!}$$

$$= \frac{x}{(2) \times 1!\, 0!} - \frac{x^3}{2^3 (2!)(1!)} + \frac{x^5}{2^5 (3!)(2!)} - \cdots \approx \sin$$



$J_{-n}(x)$, $J_n(x)$, $J_n(-x)$

$$\boxed{J_n(x) = (-1)^n J_n(-x)}$$

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$$J_{-n}(x) = \sum_{r=n}^{\infty} \frac{(-1)^r (x/2)^{2r-n}}{\Gamma(-n+r+1)\, r!}$$

- Γ is not defined at 0 or any -ve int,
- $r \neq n-1$ or $n-2$ or ...
- $\therefore$ summation starts at n

take $r-n=s$.
$r = n+s$

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{n+s} (x/2)^{n+2s}}{\Gamma(s+1)\,(n+s)!}$$

$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s (x/2)^{2s+n}}{\Gamma(s+n+1)\, s!}$$

$$= (-1)^n \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{\Gamma(n+r+1)\, r!}$$

$$\boxed{J_{-n}(x) = (-1)^n J_n(x)}$$

$$\boxed{J_n(x) = (-1)^n J_n(-x)}$$

Recurrance Relations (all 6)

1. $$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)$$

LHS: $$\frac{d}{dx}\sum_{r=0}^{\infty}\frac{(-1)^r \, x^{2r+2n}}{2^{2r+n}\,\Gamma(n+r+1)\, r!}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r \,(2)(r+n)\, x^{2r+2n-1}}{2^{2r+n}\,\Gamma(n+r+1)\, r!}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r \,(2)\,(r+n)\, x^{2r+n-1}\, x^n}{2^{2r+n}\,(n+r)\,\Gamma(n+r)\, r!}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r \,\cancel{(r+n)}\, x^{2r+(n-1)}\, x^n}{2^{2r+n-1}\,\cancel{(n+r)}\,\Gamma(n+r)\, r!}$$

$$= x^n \sum_{r=0}^{\infty}\frac{(-1)^r \, x^{(n-1)+2r}}{2^{(n-1)+2r}\,\Gamma(n-1+r+1)\, r!}$$

$$\frac{d\,(x^n J_n(x))}{dx} = x^n J_{n-1}(x)$$

2. $\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x)$

LHS: $\frac{d}{dx}\left(x^{-n} \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{\Gamma(n+r+1)\, r!}\right)$

$$= \frac{d}{dx}\left(\sum_{r=0}^{\infty} \frac{(-1)^r\, x^{2r}}{2^{n+2r}\, \Gamma(n+r+1)\, r!}\right) =$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2r)\, x^{2r-1}}{2^{n+2r}\, \Gamma(n+r+1)\, r!} = \sum_{r=0}^{\infty} \frac{(-1)^r (2r)(x^{-n})(x^{2r+n-1})}{2^{n+2r}\, \Gamma(n+r+1)\, r!}$$

$$= \sum_{r=1}^{\infty} \frac{(-1)^r\, x^{-n}\, x^{2r+n-1}}{2^{n+2r-1}\, \Gamma(n+r+1)\, (r-1)!} \qquad s = r-1,\ r = s+1$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^{s+1}\, x^{-n}\, x^{n+2s+1}}{2^{n+2s+1}\, \Gamma(n+s+1+1)\, s!}$$

$$= (-1)x^{-n} \sum_{s=0}^{\infty} \frac{(-1)^s\, \cancel{x^{-n}}\, x^{(n+1)+2s}}{2^{(n+1)+2s}\, \Gamma((n+1)+s+1)\, s!} = -x^{-n} J_{n+1}(x).$$

3. $J_n'(x) = J_{n-1} - \frac{n}{x} J_n(x)$ or $x J_n'(x) = x J_{n-1}(x) - n J_n(x)$.

LHS: $x \frac{d}{dx}\left(\sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{\Gamma(n+r+1)\, r!}\right)$

$$= x \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)\, x^{n+2r-1}}{2^{n+2r}\, \Gamma(n+r+1)\, r!}$$

$$= x\left(2\sum_{r=0}^{\infty} \frac{(-1)^r (n+r)\, x^{n+2r-1}}{2^{n+2r}\, \Gamma(n+r+1)\, r!}\right) + \sum_{r=0}^{\infty} \frac{(-1)^{r+1}\, n\, x^{n+2r}}{2^{n+2r}\, \Gamma(n+r+1)\, r!}$$

$$= x \sum_{r=0}^{\infty} \frac{(-1)^r \cancel{(n+r)}\, x^{(n-1)+2r}}{2^{(n-1)+2r} \cancel{(n+r)} \Gamma(n+r)\, r!}$$

$$- n \sum_{r=0}^{\infty} \frac{(-1)^r\, x^{n+2r}}{2^{n+2r} \Gamma(n+r+1)\, r!}$$

$$= x J_{n-1}(x) - n J_n(x) = RHS = (J_n'(x))x.$$

4. $J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$ or.

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x)$$

LHS:

$$x \frac{d}{dx}\left(\sum_{r=0}^{\infty} \frac{(-1)^r (x)^{n+2r}}{2^{n+2r} \Gamma(n+r+1)\, r!} \right)$$

$$= x \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)\, x^{n+2r-1}}{2^{n+2r} \Gamma(n+r+1)\, r!}$$

$$= n \sum_{r=0}^{\infty} \frac{(-1)^r x^{n+2r}}{2^{n+2r} \Gamma(n+r+1)\, r!} + x \sum_{r=0}^{\infty} \frac{2r\, (-1)^r x^{n+2r-1}}{2^{n+2r} \Gamma(n+r+1)\, r!}$$

$$= n J_n(x) + x \sum_{r=1}^{\infty} \frac{(-1)^r\, x^{n+2r-1}}{2^{n+2r-1} \Gamma(n+r+1)(r-1)!}$$ $s = r-1$, $r = s+1$

$$= n J_n(x) + x \sum_{s=0}^{\infty} \frac{(-1)^{s+1}\, x^{(n+1)+2s}}{2^{(n+1)+2s} \Gamma((n+1)+s+1)\, s!}$$

$$= n J_n(x) - x J_{n+1}(x) = RHS.$$

5. $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

$$\text{LHS}: \frac{d}{dx}\sum_{r=0}^{\infty} \frac{(-1)^r x^{n+2r}}{2^{n+2r}\Gamma(n+r+1)r!} = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r) x^{n+2r-1}}{2^{n+2r}\Gamma(n+r+1)r!}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (n+r) x^{n+2r-1}}{2^{n+2r}\Gamma(n+r+1)r!} + \sum_{r=1}^{\infty} \frac{(-1)^r \cancel{r}\, x^{n+2r-1}}{2^{n+2r}\Gamma(n+r+1)\cancel{r}(r-1)!}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r x^{n+2r-1}}{2\cdot 2^{n+2r-1}\Gamma(n+r)r!} + \sum_{s=0}^{\infty} \frac{(-1)^{s+1} x^{n+1+2s}}{2\cdot 2^{n+2s+1}\Gamma(n+s+2)s!}$$

$$= \frac{1}{2}J_{n-1}(x) - \frac{1}{2}J_{n+1}(x) = \text{RHS}.$$

6. $\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x)$ * no derivative!!

$$\text{LHS}: \frac{2n}{x}\sum_{r=0}^{\infty} \frac{(-1)^r x^{n+2r}}{2^{n+2r}\Gamma(n+r+1)r!} = \frac{2(n+r-r)}{x}\sum_{r=0}^{\infty} \frac{(-1)^r x^{2r+n}}{2^{n+2r} r!\,\Gamma(n+r+1)}$$

$$= \sum_{r=0}^{\infty} \frac{\cancel{(n+r)}(-1)^r x^{(n-1)+2r}}{2^{(n-1)+2r}\cancel{(n+r)}\Gamma(n+r)r!} + \sum_{r=1}^{\infty} \frac{(-1)^{r+1} x^{2r+n-1}}{2^{n-1+2r}\Gamma(n+r+1)(r-1)!}$$

$$= J_{n-1}(x) + \sum_{s=0}^{\infty} \frac{(-1)^s (1)\, x^{n+1+2s}}{2^{n+1+2s}\Gamma(n+s+2)s!} = J_{n-1}(x) + J_{n+1}(x) = \text{RHS}$$

FORMULA LIST

1. $\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$

2. $\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)$

3. $\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x)$

4. $J_n'(x) = J_{n-1} - \frac{n}{x} J_n(x)$

5. $J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$

6. $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

7. $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

8. $J_{-n}(x) = (-1)^n J_n(x)$
9. $J_n(-x) = (-1)^n J_n(x)$

n is +ve

10. $e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$

11. $\cos(x \sin\theta) = J_0 + 2J_2 \cos 2\theta + 2J_4 \cos 4\theta + \dots$

12. $\sin(x \sin\theta) = 2J_1 \sin\theta + 2J_3 \sin 3\theta + \dots$

13. $J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta$

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35. Express $J_5(x)$ in terms of J_0 and J_1

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x).$$

$$J_5(x) = \frac{8}{x} J_4(x) - J_3(x)$$

$$= \frac{8}{x}\left(\frac{6}{x} J_3(x) - J_2(x)\right) - \left(\frac{4}{x} J_2(x) - J_1(x)\right)$$

$$= \frac{8}{x}\left(\frac{6}{x}\left(\frac{4}{x} J_2(x) - J_1(x)\right) - J_2(x)\right) - \left(\frac{4}{x} J_2(x) - J_1(x)\right)$$

$$= \frac{48}{x^2}\left(\frac{4}{x} J_2(x) - J_1(x)\right) - \frac{12}{x} J_2(x) + J_1(x)$$

$$= \frac{192}{x^3} J_2(x) - \frac{48}{x^2} J_1(x) - \frac{12}{x} J_2(x) + J_1(x)$$

$$= \left(\frac{192}{x^3} - \frac{12}{x}\right)\left(\frac{2}{x} J_1(x) - J_0(x)\right) + J_1(x)\left(\frac{-48}{x^2} + 1\right)$$

$$= J_1(x)\left(\frac{384}{x^4} - \frac{24}{x^2} - \frac{48}{x^2} + 1\right) + J_0(x)\left(\frac{12}{x} - \frac{192}{x^3}\right)$$

$$= J_1(x)\left(\frac{384}{x^4} - \frac{72}{x^2} + 1\right) + J_0\left(\frac{12}{x} - \frac{192}{x^3}\right)$$

36. P.T $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ and $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$.

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{\Gamma(n+r+1)\, r!}$$

(a)

$$J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\, x^{\frac{1}{2}+2r}}{2^{1/2+2r}\,\Gamma(\frac{3}{2}+r)\, r!} = \sqrt{\frac{x}{2}} \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{2^{2r}\,\Gamma(\frac{3}{2}+r)\, r!}$$

$$= \sqrt{\frac{x}{2}}\left(\frac{1}{\Gamma(3/2)\, 0!} - \frac{(x/2)^2}{\Gamma(5/2)\, 1!} + \frac{(x/2)^4}{\Gamma(7/2)\, 2!} - \cdots\right)$$

$$= \sqrt{\frac{x}{2}}\left(\frac{1}{\Gamma(3/2)}\right)\left(\frac{1}{0!} - \frac{(x/2)^2}{3/2} + \frac{(x/2)^4}{(3/2)(5/2)\,2!} - \cdots\right)$$

$$= \sqrt{\frac{x}{2}}\left(\frac{1}{(1/2)\sqrt{\pi}}\right)\left(1 - \frac{x^2}{3\cdot 2} + \frac{x^4}{3\cdot 5\cdot 4\cdot 2} - \cdots\right)$$

$$= \sqrt{\frac{x}{2}}\left(\frac{2}{\sqrt{\pi}\,x}\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= \sqrt{\frac{2}{\pi x}}\,\sin x$$

(b) $$J_{-1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{-1/2+2r}}{\Gamma(+1/2+r)\,r!}$$

$$= \sqrt{\frac{2}{x}} \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{2r}}{\Gamma(1/2+r)\,r!}$$

$$= \sqrt{\frac{2}{x}}\left(\frac{1}{\Gamma(1/2)} - \frac{(x/2)^2}{\Gamma(3/2)\,1!} + \frac{(x/2)^4}{\Gamma(5/2)\,2!} \cdots\right)$$

$$= \sqrt{\frac{2}{x}}\left(\frac{1}{\Gamma(1/2)}\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{(3/2)(1/2)(2^4\cdot 2!)} - \cdots\right)$$

$$= \sqrt{\frac{2}{\pi x}}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$

$$= \sqrt{\frac{2}{\pi x}}\,\cos x$$

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37. Evaluate $J_{5/2}(x)$

$$\frac{2n}{x} J_n(x) = \left(J_{n-1}(x) + J_{n+1}(x)\right)$$

$$J_{n+1}(x) = J_{n-1}(x) - \frac{2n}{x} J_n(x)$$

$$J_{3/2+1}(x) = J_{3/2-1}(x) - \frac{2(3/2)}{x} J_{3/2}(x)$$

$$J_{5/2}(x) = J_{1/2}(x) - \frac{3}{x}\left(J_{3/2}x\right)$$

$$= J_{1/2}(x) - \frac{3}{x}\left(J_{-1/2}(x) - \frac{2\times 1/2}{x} J_{1/2}(x)\right)$$

$$= \sqrt{\frac{2}{\pi x}} \sin x - \frac{3}{x}\left(\sqrt{\frac{2}{\pi x}} \cos x - \frac{1}{x}\sqrt{\frac{2}{\pi x}} \sin x\right)$$

$$= \sqrt{\frac{2}{\pi x}} \sin x - \frac{3}{x}\sqrt{\frac{2}{\pi x}} \cos x + \frac{3}{x^2}\sqrt{\frac{2}{\pi x}} \sin x$$

$$= \sqrt{\frac{2}{\pi x}}\left(\sin x - \frac{3}{x}\cos x + \frac{3}{x^2}\sin x\right)$$

38. Evaluate $J_{-5/2}(x)$

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

$$J_{n-1}(x) = \frac{2n J_n(x)}{x} J_{n+1}(x)$$

$$= \frac{\cancel{2}\times 3/\cancel{2}}{x}\left(J_{3/2}(x)\right) - J_{-1/2}(x)$$

$$= \frac{3}{x}\left(\frac{\cancel{2}(-1/\cancel{2})}{x} J_{-1/2}(x) - J_{1/2}(x)\right) - J_{-1/2}(x)$$

$$= \frac{3}{x}\left(-\frac{1}{x}\sqrt{\frac{2}{\pi x}} \cos x - \sqrt{\frac{2}{\pi x}} \sin x\right) - \sqrt{\frac{2}{\pi x}} \cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left(+\frac{3}{x^2}\cos x - \frac{3}{x}\sin x - \cos x\right)$$

39. $\int J_3(x)\,dx$ $\qquad \frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x)$

$= \int x^2 - x^{-2} J_3(x)\,dx = \int -x^2 \frac{d}{dx}\left(x^{-2} J_2(x)\right) dx$

$u = -x^2$ $\qquad v = -x^{-2} J_2(x)$

$du = -2x\,dx$ $\qquad dv = \frac{d}{dx}\left(-x^2 J_2(x)\right)$

$= -\cancel{x^2}\,\cancel{x^{-2}}\, J_2(x) + \int 2x\, x^{-2} J_2(x)\,dx$

$= -J_2(x) + 2\int -x^{-1} J_2(x)\,dx$

$= -J_2(x) - 2\left(x^{-1} J_1(x)\right)$

$= -J_2(x) - 2x^{-1} J_1(x)$

$= -\left(\frac{2\times 1}{x} J_1(x) - J_0(x)\right) - \frac{2}{x} J_1(x)$

$= -\frac{4}{x} J_1(x) + J_0(x)$

40. $\int x^4 J_1(x)\,dx =$ $\qquad \frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)$

$n = 2$

$= \int x^2 x^2 J_1(x)\,dx = \int x^2 \frac{d}{dx}\left(x^2 J_2(x)\right) dx$

$= x^2\left(x^2 J_2(x)\right) - \int 2x\left(x^2 J_2(x)\right) dx$

$= x^4 J_2(x) - 2\int x^3 J_2(x)\,dx$ $\qquad \boxed{n = 3}$

$= x^4 J_2(x) - 2\, x^3 J_3(x) = \cancel{x^4 J_2(x) - 2x^3}$

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

$$n=2$$

$$= x^4 J_2(x) - 2x^3\left(\frac{4}{x} J_2(x) - J_1(x)\right)$$

$$= x^4 J_2(x) - 8x^2 J_2(x) + 2x^3 J_1(x)$$

$n=1$

$$= (x^4 - 8x^2)\left(\frac{2}{x} J_1(x) - J_0(x)\right) + 2x^3 J_1(x)$$

$$= (x^4 - 8x^2)\frac{2}{x} J_1(x) - (x^4 - 8x^2) J_0(x) + 2x^3$$

$$= (4x^3 - 16x) J_1(x) - (x^4 - 8x^2) J_0(x)$$

40. **NOTE:** $\frac{d}{dx}\left(J_0(x)\right) = J_{-1}(x) = -1\, J_1(x)$

$$\int J_1(x)\,dx = -J_0(x)$$

41. $\int x J_0^2(x)\,dx = \int J_0(x)\, \underbrace{x J_0(x)}\,dx$

$$= \int J_0(x) \frac{d}{dx}\left(x J_1(x)\right) dx = J_0(x)\, x J_1(x) - \int J_0'(x)\, x J_1(x)\,dx$$

$$= x J_0(x) J_1(x) + \int x J_1^2(x)\,dx$$

$$= x J_0(x) J_1(x) + \int -J_1(x)\, \underbrace{x J_1(x)}\,dx$$

$n=0$

$$= x J_0(x) J_1(x) + \int x J_1(x) \frac{d}{dx} J_0(x)\,dx$$

$$= x J_0(x) J_1(x) - x J_1(x) J_0(x) + \int \frac{d}{dx}(x J_1(x)) J_0 x$$

$$= \int \left(J_1(x) + x \frac{d}{dx}\left(\frac{x J_1(x)}{x}\right)\right) J_0(x)$$

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$$\cancel{\int J_1(x)\, J_0(x) + x\, J_0(x) \frac{d}{dx}\left(x J_1(x) \cdot \frac{1}{x}\right)}$$

$$\cancel{\int J_1(x)\, J_0(x) + x\, J_0(x)\left(\frac{x J_0 x}{x} + x J_1(x) \cdot -\frac{1}{x^2}\right)}$$

$$\cancel{= x J_0(x) J_1(x) + \int x J_1^2(x)\, dx}$$

$$\cancel{= x J_0(x) J_1(x) + x J_1(x)^2 dx}$$

$\longrightarrow$ start here

$$= \int x J_0^2(x)\, dx = \frac{x^2}{2} J_0^2(x) + \int \frac{x^2}{2}\, 2 J_0(x) J_1(x)$$

$$= \frac{x^2}{2} J_0^2(x) + \int x^2 J_0(x) J_1(x)\, dx$$

$n \Rightarrow 2$

$$\cancel{\frac{x^2}{2} J_0^2(x) + \int J_0(x) \frac{d}{dx}\left(x^2 J_2(x)\right) dx}$$

$$\cancel{\frac{x^2}{2} J_0^2(x) + J_0(x) x^2 J_2(x) + \int J_1(x) x^2 J_2(x)\, dx}$$

$$= \frac{x^2}{2} J_0^2(x) + \int x J_1(x)\, x J_0(x)\, dx$$

$$= \frac{x^2}{2} J_0^2(x) + \int x J_1(x) \frac{d}{dx}\left(x J_1(x)\right) dx$$

$$= \frac{x^2}{2} J_0^2(x) + \frac{(x J_1(x))^2}{2}$$

42. Verify that $y = x^n J_n(x)$ is a solution to DE

$$xy'' + (1-2n)y' + xy = 0$$

$$y' = nx^{n-1} J_n(x) + x^n J_n'(x)$$

$$= nx^{n-1} J_n(x) + x^n \left(\frac{1}{2}\right)\left(J_{n+1}(x) - J_{n-1}(x)\right)$$

$$= nx^{n-1} J_n(x) + \frac{x^n}{2}\left(J_{n+1}(x) - J_{n-1}(x)\right)$$

$$y'' = (n)(n-1)x^{n-2} J_n(x) + nx^{n-1} J_n'(x)$$
$$+ \frac{nx^{n-1}}{2} J_{n+1}(x) + \frac{x^n}{2} J_{n+1}'(x)$$
$$- \frac{nx^{n-1}}{2} J_{n-1}(x) - \frac{x^n}{2} J_{n-1}'(x)$$

$$xy'' = (n)(n-1)x^{n-1} J_n(x) + nx^n J_n'(x) + \frac{x^n n}{2} J_{n+1}(x)$$
$$+ \frac{x^{n+1}}{2} J_{n+1}'(x) - \frac{nx^n}{2} J_{n-1}(x) - \frac{x^{n+1}}{2} J_{n-1}'(x)$$

$$(1-2n)y' = (1-2n)nx^{n-1} J_n(x) + (1-2n)\frac{x^n}{2}\left(J_{n+1}(x) - J_{n-1}(x)\right)$$

$$y'' = (n)(n-1)x^{n-2} J_n(x) + nx^{n-1} J_n'(x) + nx^n J_n'(x)$$
$$+ x^n J_n''(x)$$

$$xy'' = (n^2-n)x^{n-1} J_n(x) + nx^n J_n'(x) + nx^{n+1} J_n'(x)$$
$$+ x^{n+1} J_n''(x)$$

$$(1-2n)y' = (n-2n^2)x^{n-1} J_n(x) + (1-2n)x^n J_n'(x)$$

$$xy = x^{n+1} J_n(x)$$

$$\text{LHS:} = J_n(x)\left[x^{n+1} + (\cancel{n}-2n^2)x^{n-1} + (n^2-\cancel{n})x^{n-1}\right]$$
$$+ J_n'(x)\left[(1-2n)x^n + nx^{n+1} + nx^n + (n^2-n)x^{n-1}\right]$$
$$+ J_n''(x)\left[x^{n+1}\right]$$

$$= x^{n-1}\Big(J_n(x)\,(x^2 - 2n^2 + n^2)$$

$$+ J_n'(x)\,\big((1-2n)x + nx^2 + nx + n^2 - n\big)$$

$$+ J_n''(x)\,(x^2)\Big)$$

$$= x^{n-1}\Big[\, x^2 J_n''(x) +$$

GENERATING FUNCTIONS

$1, 0, \frac{1}{2!}, 0, \frac{1}{4!}, \ldots \qquad \frac{\left(\sin\left(\frac{n\pi}{2}\right)\right)^2}{n!} \;;\; \frac{1 + (-1)^n}{2(n!)}$

$1, 1, 1, \ldots$

$1, 1, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}$

Find a function that generates Bessel

$$\ldots, J_1, J_2, J_3 \ldots = \sum_{n=-\infty}^{-\infty} J_n(x)\, t^n$$

$$\ldots J_{-3}(x)t^{-3} + J_{-2}(x)\, t^{-2} + J_{-1}(x)\, t^{-1} + J_0(x)\, t^0 + J_1(x)\, t^1 + J_2(x)\, t^2$$

$\Rightarrow$ We know $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{2r+n}}{\Gamma(n+r+1)\, r!}$

$$J_n(x) = (-1)^n J_{-n}(x)$$

$$= J_3\left(t^3 - \frac{1}{t^3}\right) + J_2\left(t^2 + \frac{1}{t^2}\right) + J_1\left(t - \frac{1}{t}\right) + J_0(\cancel{1}) + \dots$$

Generating series: $e^{x/2(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$

$$= e^{xt/2} \cdot e^{-x/2t}$$

$$= \left(1 + \frac{xt}{2} + \frac{\left(\frac{xt}{2}\right)^2}{2!} + \frac{\left(\frac{xt}{3}\right)^3}{3!} + \dots + \frac{\left(\frac{xt}{2}\right)^n}{n!} + \boxed{\frac{\left(\frac{xt}{2}\right)^{n+1}}{(n+1)!}} + \dots\right)$$

$$\left(1 - \boxed{\frac{x}{2t}} + \frac{\left(\frac{x}{2t}\right)^2}{2!} - \frac{\left(\frac{x}{2t}\right)^3}{3!} + \dots + (-1)^n \frac{\left(\frac{x}{2t}\right)^n}{n!} + \right)$$

Grouping coeff of t^n:

$$\frac{\left(\frac{x}{2}\right)^n}{n!} - \frac{\frac{x^{n+2}}{2^{n+2}}}{(n+1)!} + \frac{\left[\frac{x}{2}\right]^{n+4}}{(n+2)!\,2!} = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{\Gamma(n+r+1)\, r!}$$

$$= J_n(x)$$

Grouping co-eff of t^{-n}

$$\frac{(-1)^n \left(\frac{x}{2}\right)^n}{n!} + \frac{(-1)^{n+1} \left(\frac{x}{2}\right)^{n+1}}{(n+1)!\,\cancel{1}!} + \dots = (-1)^n \left[\frac{(x/2)^n}{n!} - \frac{(x/2)^{n+1}}{(n+1)!\,1!}\right]$$

$$= (-1)^n J_n(x)$$

$$= J_{-(n)}\, x.$$

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The generating function of Bessel function:

$$e^{\frac{x}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

☆43. Establish the Jacobi Series no -ve J

$$\cos(x\cos\theta) = J_0 - 2J_2\cos 2\theta + 2J_4\cos 4\theta - \ldots$$

$$\sin(x\cos\theta) = 2\left[J_1\cos\theta - J_3\cos 3\theta + \ldots\right]$$

$$\cancel{\cos\phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \ldots}$$

$$\cancel{\cos(x\cos\theta) =}$$

Generating function $= e^{\frac{x}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n t^n$

$$= \ldots J_{-2}t^{-2} + J_{-1}t^{-1} + J_0 t^0 + J_1 t^1 + J_2 t^2 + \ldots$$

$$= J_0(1) + J_1\left(t - \frac{1}{t}\right) + J_2\left(t^2 + \frac{1}{t^2}\right) + J_3\left(t^3 - \frac{1}{t^3}\right) + \ldots$$

$$t = \cos\theta + i\sin\theta = e^{i\theta}$$

$$\frac{1}{t} = \cos\theta - i\sin\theta = e^{-i\theta} \qquad t - \frac{1}{t} = 2i\sin$$

$$t^n + \frac{1}{t^n} = 2\cos n\theta \;;\; t^n - \frac{1}{t^n} = 2i\sin n\theta$$

$$e^{\frac{x}{\cancel{2}}(\cancel{2}i\sin\theta)} = e^{xi\sin\theta}$$

$$= J_0 + J_1(2i\sin\theta) + J_2(2\cos 2\theta) + J_3(2i\sin 3\theta) + \ldots$$

$e^{i\sin\theta\, x} = \cos(x\sin\theta) + i\sin(x\sin\theta)$.

Grouping all real and imaginary

$\cos(x\sin\theta) = J_0 + J_2(2\cos 2\theta) + 2J_4(\cos 4\theta) + \dots$

$i\sin(x\sin\theta) = 2i(J_1\sin\theta + J_3\sin 3\theta + \dots)$

$\cos(x\sin\theta) = J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \dots$

$\sin(x\sin\theta) = 2J_1\sin\theta + 2J_3\sin 3\theta + 2J_5\sin 5\theta + \dots$

Replace $\theta \to \pi/2 - \theta$.

$\cos(x\cos\theta) = J_0 - 2J_2\cos 2\theta + 2J_4\cos 4\theta - \dots$
$\sin(x\cos\theta) = 2J_1\cos\theta - 2J_3\cos 3\theta + 2J_5\cos 5\theta + \dots$

44. Prove that $J_0^2 + 2J_1^2 + 2J_2^2 + \dots = 1$.

G.F $e^{\frac{x}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n t^n$

$\cos^2(x\sin\theta) + \sin^2(x\sin\theta) = (J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \dots)^2 +$
$(2J_1\sin\theta + 2J_3\sin 3\theta + 2J_5\sin 5\theta)^2$

$1 = J_0^2 + 4J_2^2\cos^2 2\theta + 4J_4^2\cos^4\theta + \dots + 4J_1^2\sin^2\theta + 4J_3^2\sin^3 3\theta + \dots$

We know $\int_0^{\pi} \cos m\theta \cos n\theta = \begin{cases} \pi/2, & m = n \\ 0, & m \neq n \end{cases}$

$= \int_0^{\pi} \sin m\theta \sin n\theta$

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Integrating both sides from 0 to π

$$\int_0^{\pi} d\theta = \int_0^{\pi} J_0^2\, d\theta + \int_0^{\pi} 2J_2^2 (2\cos 2\theta)^2 d\theta + \int_0^{\pi} 2J_4^2 (2\cos 4\theta)^2 d\theta$$

$$+ \dots + \int_0^{\pi} (2J_1 \sin \theta)^2 + \int_0^{\pi} (2J_3 \sin 3\theta)^2 + \dots$$

$$\Rightarrow \pi = J_0^2 \pi + 2J_2^2 \frac{\pi}{\cancel{2}} + 2J_4^2 \pi + \dots + 2J_1^2 \pi + 2J_3^2 \pi$$

$$\boxed{1 = J_0^2 + 2J_2^2 + 2J_3^2 + 2J_4^2 + \dots}$$

Hence proved.

45. Prove that (i) $2[J_1 - J_3 + J_5 - \dots] = \sin x$

(ii) $J_0 - 2J_2 + 2J_4 - 2J_6 + \dots = \cos x$

(iii) $1 = J_0 + 2J_2 + 2J_4 + \dots$

(i) $\sin(x \sin\theta) = 2J_1 \sin\theta + 2J_3 \sin 3\theta + 2J_5 \sin 5\theta + \dots$

if $\theta = \pi/2$

$\sin x = 2J_1 - 2J_3 + 2J_5 - \dots$

(ii) $\cos(x\sin\theta) = J_0 + 2J_2 \cos 2\theta + 2J_4 \cos 4\theta + \dots$

if $\theta = \pi/2$

$\cos x = J_0 - 2J_2 + 2J_4 - 2J_6 + \dots$

(iii) $\cos(x\sin\theta) = J_0 + 2J_2 \cancel{\cos} \cos 2\theta + 2J_4 \cos 4\theta$

if $\theta = 0$

$\cos 0 = 1 = J_0 + 2J_2 + 2J_4 + \dots$

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INTEGRAL FORM OF BESSEL FUNCTION

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos n\theta \cos(x\sin\theta) + \sin n\theta \sin(x\sin\theta)\, d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos n\theta\left(J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \dots\right) + \sin n\theta\left(2J_1\sin\theta + 2J_3\sin 3\theta + \dots\right) d\theta.$$

if n is even, $= \frac{1}{\pi}\int_0^{\pi} 2J_n\cos^2 n\theta\, d\theta = J_n(x)$.

if n is odd, $= \frac{1}{\pi}\int_0^{\pi} 2J_n\sin^2 n\theta\, d\theta = J_n(x)$

if $n = 0$, $= \frac{1}{\pi}\int_0^{\pi} J_0\, d\theta = J_0(x)$

Equations Reducible to Bessel's D.E.

1) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2x^2 - n^2)y = 0.$

$t = \lambda x \Rightarrow x = \frac{t}{\lambda} \Rightarrow dt = \lambda dx \Rightarrow \frac{dt}{dx} = \lambda$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \lambda\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\lambda\frac{dy}{dt}\right) = \frac{d}{dt}\left(\lambda\frac{dy}{dt}\right)\cdot\frac{dt}{dx} = \lambda\frac{d^2y}{dt^2}\cdot\lambda$$

$$\frac{d^2y}{dx^2} = \lambda^2\frac{d^2y}{dt^2}$$

$$\frac{t^2}{\cancel{\lambda^2}} \cancel{x^2} \frac{d^2y}{dt^2} + \frac{t}{\cancel{x}} \cancel{x} \frac{dy}{dt} + (t^2-n^2)y = 0$$

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2-n^2)\,y = 0$$

$$y_0 = C_1 J_n(t) + C_2 J_{-n}(t)$$

solution: $\boxed{y = C_1 J_n(\lambda x) + C_2 J_{-n}(\lambda x)}$

Orthogonality of Bessel Functions

Two functions $f(x)$ and $g(x)$ are said to be orthogonal ~~ed~~ over the interval a to b if

$$\int_a^b f(x)\, g(x)\, dx = 0$$

eg: $\int_0^{\pi} \sin x \cos x \, dx$

Sometimes, a weighted function is added to make the integral 0 (two non-orthogonal functions)

$$\int_a^b W(x) f(x) g(x)\, dx$$

For Bessel functions:

$$\int_0^1 x J_n(\alpha x) J_n(\beta x) = \begin{cases} 0, & \alpha \neq \beta \\ \frac{1}{2}\left[J_{n+1}(\alpha)\right]^2_b \cancel{\alpha}, & \alpha = \beta \end{cases}$$

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If α and β are the roots of the equation $J_n(x) = 0$, then

$$\int_0^1 x\, J_n(\alpha x)\, J_n(\beta x)\, dx = \begin{cases} 0, & \alpha \neq \beta \\ \frac{1}{2}[J_n(\alpha)]^2, & \alpha = \beta \end{cases}$$

$\because J_n(\alpha) = 0 = J_n(\beta)$

$J_n(\alpha x)$ sol. of DE $x^2 u'' + x u' + (\alpha^2 x^2 - n^2) u = 0$ — (1)

$\therefore J_n(\alpha x) = u$

$J_n(\beta x)$ sol. of DE $x^2 v'' + x v' + (\beta^2 x^2 - n^2) v = 0$ — (2)

$\therefore J_n(\beta x) = v$

Multiplying (1) with v/x & (2) with u/x.

$$x u'' v + u' v + \alpha^2 x u v - n^2 \frac{uv}{x} = 0$$

$$-\left(x v'' u + v' u + \beta^2 x u v - n^2 \frac{uv}{x} = 0\right)$$

$$\underbrace{x(u''v - uv'') + (u'v - uv')}\, + (\alpha^2 - \beta^2)(xuv) = 0.$$

$$\frac{d}{dx}\left(x(u'v - uv')\right) = x(u''v + u'v' - u'v' - uv'') + (u'v - uv')$$

$$= \frac{d}{dx}\left(x(u'v - uv')\right) + (\alpha^2 - \beta^2)(xuv) = 0$$

$$\int_0^1 xuv = \int_0^1 \frac{\frac{d}{dx}\left(x(u'v - uv')\right) dx}{\beta^2 - \alpha^2}$$

$$\int_0^1 xuv = \left[\frac{x(u'v-uv')}{\beta^2-\alpha^2}\right]_0^1$$

$$= \int_0^1 xuv = \left[\frac{x\left(J_n'(\alpha x)J_n(\beta x) - J_n(\alpha x)J_n'(\beta x)\right)}{\beta^2-\alpha^2}\right]_0^1$$

$$= \frac{J_n'(\alpha)J_n(\beta) - J_n(\alpha)J_n'(\beta)}{\beta^2-\alpha^2}$$

<u>case(i)</u> $\alpha \neq \beta$ $\quad (J_n(\alpha) = J_n(\beta) = 0)$

$$\int_0^1 x\, J_n(\alpha x)J_n(\beta x)\,dx = 0$$

<u>case(ii)</u> $\alpha = \beta$

$$\lim_{\beta\to\alpha} \frac{\alpha\, J_n'(\alpha)\, J_n(\beta)}{\beta^2-\alpha^2} \quad \left[\left(\frac{0}{0}\right)\right]$$

L'Hopital's Rule

$$\lim_{\beta\to\alpha} \frac{\alpha\, J_n'(\alpha)\, J_n'(\beta)}{2\beta} = \frac{\cancel{\alpha}\,[J_n'(\alpha)]^2}{2\cancel{\alpha}}$$

$$= \frac{[J_n'(\alpha)]^2}{2} = \frac{1}{2}[J_{n+1}(\alpha)]^2$$

$$\left[J_n'(\alpha) = \frac{n}{\alpha}J_n(\alpha) - J_{n+1}(\alpha)\right]$$